Lecture 2 (Predicates and Quantifiers)

Note that the sentence "$P(x) := x + 2 = 2x$" is not a proposition. However, if we assign a value for x then it becomes a proposition. As for each value of x the sentence is either true or false. Thus the sentence can be treated as a function for which input is a value of x and the output is a proposition. Such sentence is an example of predicate or a propositional function. We define it more precise way as follows:

Predicate or Propositional function: Let A be a given set. A propositional function defined on A is an expression $P(x)$ which has the property that $P(a)$ is true or false for each $a \in A$. That is $P(x)$ becomes a statement whenever x is replaced by any value $a \in A$.

The set A is called the domain of $P(x)$, and the set T_p of all elements of A for which $P(a)$ is true is called the truth set of $P(x)$. In other words, $T_p = \{x : x \in A, P(x) \text{ is true}\}$

Example: Find the truth set T_p of each propositional function $P(x)$ defined on the set $\mathbb{N}$.

1. Let $P(x)$ be "$x + 5 > 1$". Then $T_p = \{x : x \in \mathbb{N}, x + 5 > 1\} = \mathbb{N}$.
2. Let $P(x)$ be "$x + 2 > 7$". Then $T_p = \{x : x \in \mathbb{N}, x + 2 > 7\} = \{6, 7, 8, \ldots\}$ consists of all integers greater than 5.
3. Let $P(x)$ be "$x + 5 < 3$". Then $T_p = \{x : x \in \mathbb{N}, x + 5 < 3\} = \emptyset$.

Remark: The above example shows that if $P(x)$ is a propositional function defined on a set A then $P(x)$ could be true for all $x \in A$, for some $x \in A$ or for no $x \in A$. In the next paragraph, we discuss this quantifiers related notion to such proposition function.

A word which is usually used before noun to express the quantity of object is called quantifier. Here we discuss few quantifiers which are used in propositional functions.

Universal Quantifier:

Let $P(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)\, P(x) \quad \text{or} \quad \forall x\, P(x)$$

which reads as "for every x in A, $P(x)$ is true statement. The symbol $\forall$ which reads "for all" or "for every" is called universal quantifier. In this case $T_p = A$ (the entire domain).

Existential Quantifier: Let $P(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)\, P(x) \quad \text{or} \quad \exists x\, P(x)$$

which reads as "there exists x in A such that $P(x)$ is true statement. The symbol $\exists$ which reads "there exists" or "for some" or "for at least one" is called existential quantifier. In this case $T_p \neq \emptyset$.

Negation of Quantified Statements

Consider the statement "All maps are linear". Its negation is either of the following equivalent statements:

"It is not the case that that all maps are linear"

"There exists at least one map which is not linear".

Symbolically, let S denote the set of all maps. Then the above negation can be written as

$$\neg(\forall x \in S)\,(x \text{ is linear}) \equiv (\exists x \in S)\,(x\,\text{is}\,\text{not}\,\text{linear}).$$

Or when $P(x)$ denotes " x is linear",

$$\neg(\forall x \in S)\,P(x) \equiv (\exists x \in S)\neg P(x) \quad \text{or} \quad \neg\forall x\,P(x) \equiv \exists x\,\neg P(x).$$

Thus we have:

1. $\neg(\forall x \in S)\,P(x) \equiv (\exists x \in S)\neg P(x)$
2. $\neg(\exists x \in S)\,P(x) \equiv (\forall x \in S)\neg P(x)$

The above rules for negations for quantifiers are called De Morgan's laws for quantifiers.

Example: What are the negations of the statements $\forall x\ (x^2 > x)$ and $\exists x\ (x^2 = 2)$?
Solution: The negation of $\forall x\ (x^2 > x)$ is the statement $\neg\forall x\ (x^2 > x)$, which is equivalent to $\exists x\ \neg(x^2 > x)$, that is, $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\ \exists x(x^2 = 2)$, which is equivalent to $\forall x\ \neg(x^2 = 2)$, that is, $\forall x(x^2 \neq 2)$.

Example: Show that $\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$.

Solution: By De Morgan's law for universal quantifiers, we know that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(\neg(P(x) \to Q(x)))$ are logically equivalent. Since $P(x) \to Q(x) \equiv \neg P(x) \vee Q(x)$, it follows that $\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$.

Nested Quantifiers: Two quantifiers are nested if one is within the scope of the other.

Example: The statement

$$\forall x \forall y(x + y = y + x)$$

says that $x+y = y+x$ for all real numbers x and y. This is the commutative law for addition of real numbers.

The statement

$$\forall x\ \exists y(x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse.

Similarly, the statement

$$\forall x\ \forall y \forall z\ (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers.

Quantifications of two variables

Statement	When True?	When False (or Negation of True)?
$\forall x\ \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y	There is a pair x, y for which $P(x, y)$ is false
$\forall x\ \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true	There is an x such that $P(x, y)$ is false for every y.
$\exists x\ \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x\ \exists y P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Example: We can express that a function $f : X \to Y$ is one-to-one using quantifiers as

$$\forall a\ \forall b\big(f(a) = f(b) \to a = b\big).$$

Example: A function $f : X \to Y$ is onto if

$$\forall y\ \exists x\ (f(x) = y).$$

Thus f is not one-to-one if there is a pair a, b for which $f(a) = f(b) \to a \neq b$.

Example: Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.
Solution: Recall that the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\exists L \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0\ \forall x\ \Big(0 < |x - a| < \delta \to |f(x) - L| < \epsilon\Big).$$

Thus negation of above is:

$$\forall L \in \mathbb{R}\ \exists \epsilon > 0\ \forall \delta > 0\ \exists x\ \Big(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon\Big).$$

Example from "A basic course in Real Analysis by S Kumaresan": Suppose we have a sentence: "In each tree in the orchard, we can find a branch in which all the leaves are green".

Let us convert the above sentence as a mathematical sentence: Let T denote the set of all trees in the orchard. Let $t \in T$ be a tree. Let B_t denote the set of all branches of the tree t. Let $b \in B_t$ be a branch of tree t. Let L_b denote the set of all leaves on the branch b. Then the above sentence can be written as:

$\forall t \in T \, \exists \, b \in B_t \, \forall \, l \in L_b, l$ is green. The negation is:

$$\neg(\forall t \in T \, \exists \, b \in B_t \, \forall \, l \in L_b, l \text{ is green}) \equiv \exists \, t \in T \, \forall \, b \in B_t \, \exists \, l \in L_b, l \text{ is not green.}$$